Light Perturbation and Detection of Gravitational Waves via Pulsar Timing Arrays

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Outline

1. Light perturbed by GWs

2. Perturbation of light and delay of photon transit time

3. Application - Pulsar Timing Array (PTA)

4. Conclusions and discussion

— Motivation —

Light interacts with GWs. From a general relativistic perspective, this can be viewed as a perturbation of light due to GWs; i.e., light is perturbed as GWs pass through space in which it travels. We address the issue how light is perturbed in the presence of GWs; for a general situation with arbitrary $K^\mu = \left(c^{-1}\omega_{\rm e}, \mathbf{K}\right)$ and $k^\mu = \left(c^{-1}\omega_{\rm g}, \mathbf{k}\right)$. Primarily, our analysis focuses on:

– Solving Maxwell's equations in a spacetime perturbed by GWs.

– Identifying a perturbation of light with a delay of the photon transit time.

– Applying the above principle to the detection of GWs via a PTA.

1. Light perturbed by GWs

(1) Spacetime geometry perturbed by GWs

Suppose that our gravitational waves propagate along the z'-axis while being polarized in the $x'y'$-plane:

$$
\begin{aligned}
h_{ij}^{+} &= h_{+}\left(e_i^{x'}\otimes e_j^{x'} - e_i^{y'}\otimes e_j^{y'}\right)\exp\left[\mathrm{i}\left(kz'-\omega_{\mathrm{g}}t\right)\right],\\
h_{ij}^{\times} &= h_{\times}\left(e_i^{x'}\otimes e_j^{y'} + e_i^{y'}\otimes e_j^{x'}\right)\exp\left[\mathrm{i}\left(kz'-\omega_{\mathrm{g}}t-\pi/2\right)\right]\ ;\quad \omega_{\mathrm{g}} = ck.
\end{aligned}
$$

Then the spacetime geometry reads in the coordinates (t, x', y', z') (*GW frame*):

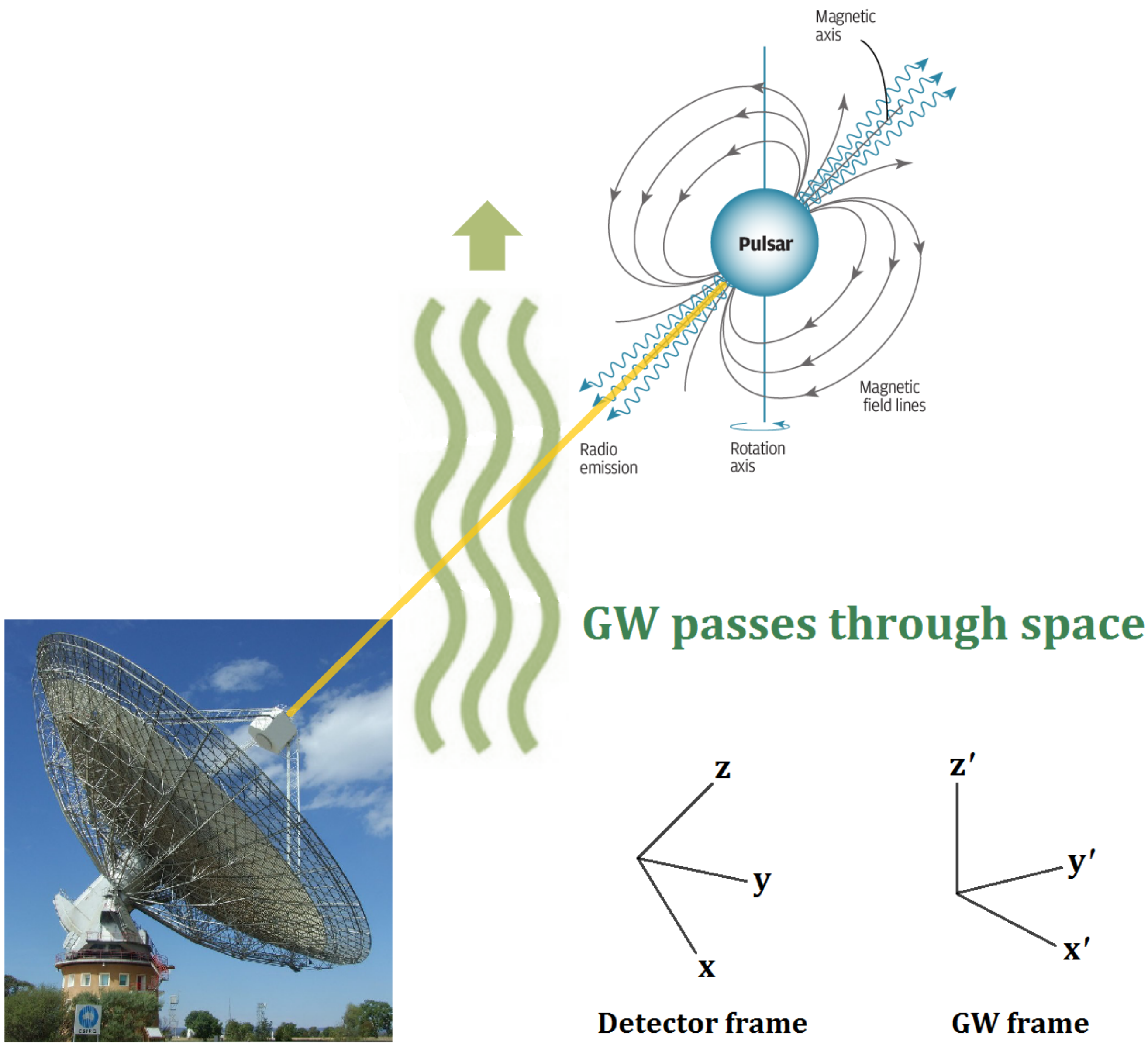
$$
ds^2 = -c^2dt^2 + \left[1+\Re\left(h_{+}e^{\mathrm{i}\left(kz'-\omega_{\mathrm{g}}t\right)}\right)\right]dx'^2 + 2\Re\left(h_{\times}e^{\mathrm{i}\left(kz'-\omega_{\mathrm{g}}t-\pi/2\right)}\right)dx'dy' + \left[1-\Re\left(h_{+}e^{\mathrm{i}\left(kz'-\omega_{\mathrm{g}}t\right)}\right)\right]dy'^2 + dz'^2.
$$

By means of Euler angles, we express

$$
\mathbf{x}' = \mathbf{R}\left(\psi,\theta,\phi\right)\mathbf{x},
$$

with

$$
\begin{aligned}
\mathbf{R}\left(\theta,\phi,\psi\right) &= \begin{bmatrix} \cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} \cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix} \cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\\
&= \begin{bmatrix} \cos\psi\cos\theta\cos\phi-\sin\psi\sin\phi & \cos\psi\cos\theta\sin\phi+\sin\psi\cos\phi & -\cos\psi\sin\theta\\ -\sin\psi\cos\theta\cos\phi-\cos\psi\sin\phi & -\sin\psi\cos\theta\sin\phi+\cos\psi\cos\phi & \sin\psi\sin\theta\\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\end{bmatrix}.
\end{aligned}
$$


Magnetic
axis
Pulsar
Magnetic
field lines
Radio
emission
Rotation
axis
GW passes through space
z
y
x
Detector frame
z′
y′
x′
GW frame

Then the spacetime geometry is expressed in the coordinates (t,x,y,z) (*detector frame*):

$$
\begin{aligned}
ds^2 \;=\; & -c^2dt^2+\Big\{1+\left[\cos(2\psi)\left(\cos^2\theta\cos^2\phi-\sin^2\phi\right)-2\sin(2\psi)\cos\theta\cos\phi\sin\phi\right]H_+ \\
& +\left[-\sin(2\psi)\left(\cos^2\theta\cos^2\phi-\sin^2\phi\right)-2\cos(2\psi)\cos\theta\cos\phi\sin\phi\right]H_\times\Big\}\,dx^2 \\
& +\Big\{\left[2\cos(2\psi)\left(1+\cos^2\theta\right)\cos\phi\sin\phi+2\sin(2\psi)\cos\theta\left(2\cos^2\phi-1\right)\right]H_+ \\
& +\left[-2\sin(2\psi)\left(1+\cos^2\theta\right)\cos\phi\sin\phi+2\cos(2\psi)\cos\theta\left(2\cos^2\phi-1\right)\right]H_\times\Big\}\,dxdy \\
& +\{[-2\cos(2\psi)\cos\theta\sin\theta\cos\phi+2\sin(2\psi)\sin\theta\sin\phi]\,H_+ \\
& +[2\sin(2\psi)\cos\theta\sin\theta\cos\phi+2\cos(2\psi)\sin\theta\sin\phi]\,H_\times\}\,dxdz \\
& +\Big\{1+\left[\cos(2\psi)\left(\cos^2\theta\sin^2\phi-\cos^2\phi\right)+2\sin(2\psi)\cos\theta\cos\phi\sin\phi\right]H_+ \\
& +\left[-\sin(2\psi)\left(\cos^2\theta\sin^2\phi-\cos^2\phi\right)+2\cos(2\psi)\cos\theta\cos\phi\sin\phi\right]H_\times\Big\}\,dy^2 \\
& +\{[-2\cos(2\psi)\cos\theta\sin\theta\sin\phi-2\sin(2\psi)\sin\theta\cos\phi]\,H_+ \\
& +[2\sin(2\psi)\cos\theta\sin\theta\sin\phi-2\cos(2\psi)\sin\theta\cos\phi]\,H_\times\}\,dydz \\
& +\Big\{1+\left[\cos(2\psi)\sin^2\theta\right]H_++\left[-\sin(2\psi)\sin^2\theta\right]H_\times\Big\}\,dz^2,
\end{aligned}
$$

with

$$
\begin{aligned}
H_+ &\equiv \Re\left(h_+\exp\left[\mathrm{i}\left(kx\sin\theta\cos\phi+ky\sin\theta\sin\phi+kz\cos\theta-\omega_{\rm g}t\right)\right]\right), \\
H_\times &\equiv \Re\left(h_\times\exp\left[\mathrm{i}\left(kx\sin\theta\cos\phi+ky\sin\theta\sin\phi+kz\cos\theta-\omega_{\rm g}t-\pi/2\right)\right]\right).
\end{aligned}
$$

[*N.B.*] For $\psi\to\psi+\pi/4$, $[\cos(2\psi)(\cdots)+\sin(2\psi)(\cdots)]h_+\to[-\sin(2\psi)(\cdots)+\cos(2\psi)(\cdots)]h_\times$**: spin-2 tensor modes of** $+$ **and** $\times$ **polarizations.**

(2) Light rays propagating through perturbed spacetime

Our light (electromagnetic radiation) can be described by Maxwell's equations defined in curved (perturbed) spacetime:

$$\Box A^{\mu} - R^{\mu}{}_{\nu} A^{\nu} = 0.$$

However, it turns out

$$R_{ij} = \mathcal{O}\left(h^2\right),$$

and hence

$$\Box A^{i} = \mathcal{O}\left(h^2\right).$$

We recast the LHS,

$$\Box A^{i} = \Box_{\rm o}\left(A^{i}_{\rm o} + \delta A^{i}_{[h]}\right) + \Box_{[h]}\left(A^{i}_{\rm o} + \delta A^{i}_{[h]}\right) + \mathcal{O}\left(h^2\right) \;\; ; \;\; \Box_{\rm o} \equiv -\frac{\partial^2}{c^2 \partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

in order to obtain a decomposition solution by means of *perturbation*:

$$A^{i} = A^{i}_{\rm o} + \delta A^{i}_{[h]} + \mathcal{O}\left(h^2\right),$$

with the zeroth-order solution from

$$\Box_{\rm o} A^{i}_{\rm o} = 0 \text{ (unperturbed)},$$

and the first-order solution from

$$\Box_{\rm o} \delta A^{i}_{[h]} = -\Box_{[h]} A^{i}_{\rm o} \text{ (first order in } h\text{)}.$$

[Solution] for a general configuration with both light and GWs propagating in *arbitrary* directions

E.g., the radio emission from a pulsar can be approximately modeled as linearly polarized light. Then to first order in h, the total solution is given by

$$A^i_{\text{total}}(t,\mathbf{x}) = A^i_{\text{o}}(t,\mathbf{x}) + \delta A^i_{[h]}(t,\mathbf{x}),$$

where the zeroth-order solution is

$$A^i_{\text{o}}(t,\mathbf{x}) \;=\; \left(-\frac{K_y}{\sqrt{K_x^2+K_y^2}}\delta^i_x + \frac{K_x}{\sqrt{K_x^2+K_y^2}}\delta^i_y\right)\mathcal{A}\exp\left[\mathrm{i}\left(\mathbf{K}\cdot\mathbf{x}-\omega_{\text{e}}t\right)\right],$$

and the first-order solution is

$$\delta A^i_{[h]}(t,\mathbf{x}) = 2\left(\omega_{\text{e}}/\omega_{\text{g}}\right)A^i_{\text{o}}(t,\mathbf{x})\,\mathcal{H}(t,\mathbf{x};\mathbf{K},\mathbf{k}),$$

with

$$\begin{aligned}\mathcal{H}(t,\mathbf{x};\mathbf{K},\mathbf{k}) \;\equiv\;& h_+\mathcal{F}_+(\phi,\theta,\psi;\mathbf{K})\cos\left(kx\sin\theta\cos\phi+ky\sin\theta\sin\phi+kz\cos\theta-\omega_{\text{g}}t\right)\\ &-h_\times\mathcal{F}_\times(\phi,\theta,\psi;\mathbf{K})\sin\left(kx\sin\theta\cos\phi+ky\sin\theta\sin\phi+kz\cos\theta-\omega_{\text{g}}t\right),\end{aligned}$$

and

$$\mathcal{F}_+(\phi,\theta,\psi;\mathbf{K}) \equiv \frac{\cos^2\gamma_2\cos(2\psi)-2\cos\gamma_2\sin\theta_\star\sin(\phi-\phi_\star)\sin(2\psi)}{2\left(1-\cos\gamma_1\right)},\quad \mathcal{F}_\times(\phi,\theta,\psi;\mathbf{K}) = \mathcal{F}_+(\phi,\theta,\psi-\pi/4;\mathbf{K}),$$

with

$$\cos\gamma_1 \equiv \cos\theta\cos\theta_\star+\sin\theta\sin\theta_\star\cos(\phi-\phi_\star),\quad \cos\gamma_2 \equiv \sin\theta\cos\theta_\star-\cos\theta\sin\theta_\star\cos(\phi-\phi_\star),$$

and

$$\sin\theta_\star\cos\phi_\star = K_x/K,\quad \sin\theta_\star\sin\phi_\star = K_y/K,\quad \cos\theta_\star = K_z/K.$$

[*N.B.*] **The full perturbation solution is** $\delta A^i_{[h]} \sim \mathcal{O}(h) + (\omega_{\rm e}/\omega_{\rm g})\,\mathcal{O}(h)$**. However, practically,** $\omega_{\rm e} \gg \omega_{\rm g}$**, and therefore** $(\omega_{\rm e}/\omega_{\rm g})\,\mathcal{O}(h)$ **is the only meaningful piece to take and remains in the** ***geometrical optics approximation*****; e.g.,** $\omega_{\rm e}/\omega_{\rm g} \sim 10^9$ **to** 10^{14} **for LIGO,** 10^{12} **to** 10^{19} **for LISA,** 10^{14} **to** 10^{17} **for PTA etc.**

[*N.B.*] **For** $\psi \to \psi - \pi/4$, $h_+\mathcal{F}_+ \sim h_+[(\cdots)\cos(2\psi) + (\cdots)\sin(2\psi)] \to h_\times\mathcal{F}_\times \sim h_\times[(\cdots)\sin(2\psi) - (\cdots)\cos(2\psi)]$**: spin-2 tensor modes of** $+$ **and** $\times$ **polarizations.**

2. Perturbation of light and delay of photon transit time

Suppose light propagates along the direction of $\mathbf{K} = (K_x, K_y, K_z) = (0, 0, -K)$ (e.g., PTA). As $K_z = -K < 0$, light propagates along $-z$ direction; i.e., from the sky towards the earth. The perturbed light can be expressed by the electric field:

$$\begin{aligned} E^i_{\rm total}(t,0,0,z) &= -c^{-1}(\partial/\partial t) A^i_{\rm total}(t,0,0,z) \\ &= E^i_{\rm o}(t,0,0,z) + \delta E^i_{[h]}(t,0,0,z) . \end{aligned}$$

Starting at $(t, z) = (t_0, L)$, the propagation path can be written as $z = L - c(t - t_0)$ for $t_0 \le t \le t_0 + T$, with $L = cT$. Then we find

$$\left.\frac{\delta E^i_{[h]}}{E^i_{\rm o}}\right|_{z=0} - \left.\frac{\delta E^i_{[h]}}{E^i_{\rm o}}\right|_{z=L} = \frac{\omega_{\rm e}}{\omega_{\rm g}}(h_+ F_+ + {\rm i} h_\times F_\times)\{1 - \exp[{\rm i}kL(1 + \cos\theta)]\}\exp[-{\rm i}(kL + \omega_{\rm g} t_0)],$$

where

$$\begin{aligned} F_+ &\equiv \mathcal{F}_+(\theta_\star \to 3\pi/2) = \sin^2(\theta/2)\cos(2\psi), \\ F_\times &\equiv \mathcal{F}_\times(\theta_\star \to 3\pi/2) = \sin^2(\theta/2)\sin(2\psi), \end{aligned}$$

are ***antenna patterns*** for + and × polarization states.

[*N.B.*] For $\psi \to \psi - \pi/4$, $h_+ F_+ \sim h_+ \cos(2\psi) \to h_\times F_\times \sim h_\times \sin(2\psi)$: **spin-2 tensor modes** of + and × **polarizations**.

From the null geodesic condition $ds^2 = 0$, a delay of the photon transit time can be expressed as

$$\begin{aligned}\frac{\delta T_{[h]}}{T} &= \frac{1}{2cT}\int_L^0 h_{zz}\left(t_0, 0, 0, z\right) dz + \mathcal{O}\left(h^2\right) \\ &\simeq -\frac{\mathrm{i}}{kL}\left(h_+ F_+ + \mathrm{i} h_\times F_\times\right)\left\{1 - \exp\left[\mathrm{i}kL\left(1 + \cos\theta\right)\right]\right\}\exp\left[-\mathrm{i}\left(kL + \omega_{\mathrm{g}} t_0\right)\right]\ ; \quad T = L/c.\end{aligned}$$

Then we establish a relation:

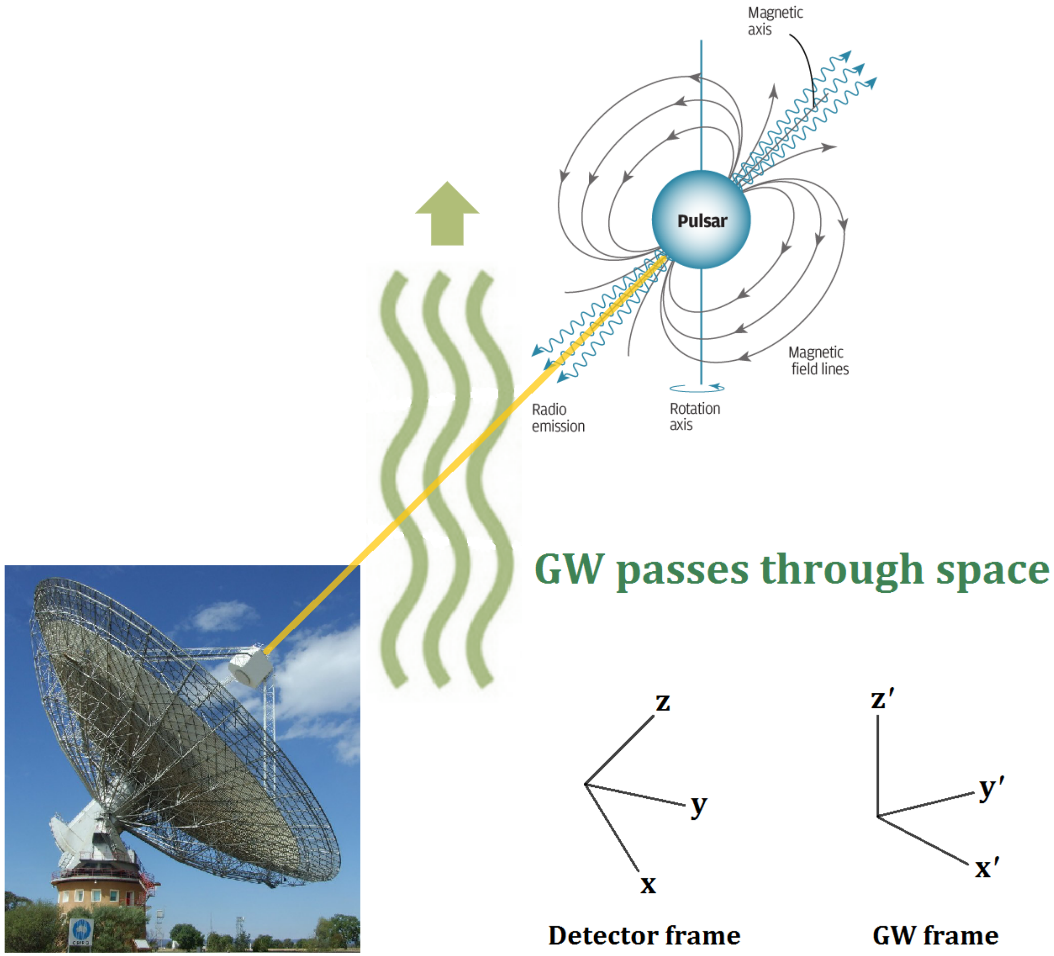
$$\frac{\delta T_{[h]}}{T} \simeq \mathcal{N}\left[\left.\frac{\delta E^i_{[h]}}{E^i_{\mathrm{o}}}\right|_{z=0} - \left.\frac{\delta E^i_{[h]}}{E^i_{\mathrm{o}}}\right|_{z=L}\right]\ ; \quad \mathcal{N} = \left(\mathrm{i}\omega_{\mathrm{e}} T\right)^{-1} = \left(\mathrm{i}KL\right)^{-1}.$$

In general, for light with *arbitrary* $\mathbf{K} = (K_x, K_y, K_z)$,

$$\begin{aligned}\frac{\delta T_{[h]}}{T} &\simeq \mathcal{N}\left[\left.\frac{\delta E^i_{[h]}}{E^i_{\mathrm{o}}}\right|_{\mathrm{earth}} - \left.\frac{\delta E^i_{[h]}}{E^i_{\mathrm{o}}}\right|_{\mathrm{sky}}\right]\ ; \quad \begin{array}{l}\mathrm{earth} = (0,0,0)\,, \\ \mathrm{sky} = \left(-L\sin\theta_\star\cos\phi_\star, -L\sin\theta_\star\sin\phi_\star, -L\cos\theta_\star\right)\end{array} \\ &\simeq -\frac{\mathrm{i}}{kL}\left(h_+ \mathcal{F}_+ + \mathrm{i} h_\times \mathcal{F}_\times\right)\left\{1 - \exp\left[\mathrm{i}kL\left(1 - \cos\gamma_1\right)\right]\right\}\exp\left[-\mathrm{i}\left(kL + \omega_{\mathrm{g}} t_0\right)\right].\end{aligned}$$

That is, a perturbation of light due to GWs is physically *equivalent* to a delay of the photon transit time: the former is described by Maxwell's equations, and the latter by the null geodesic equation.

3. Application - Pulsar Timing Array (PTA)



In order to measure pulse arrival time of a pulsar, one can arrange a detector (radio telescope) to receive photons emitted from the pulsar. A pulsar can serve as an astronomical clock of excellent precision, with the constancy of the measured pulse frequency ν_o. However, with GWs passing through our space, the measured frequency $\nu(t)$ will vary slightly. Then the effects of GWs can be determined from the variation of the frequency, $[\nu_o - \nu(t)]/\nu_o$.

(1) GW signal readout and response function for PTA

Instead of the frequency ν, consider the elapse $\tau = \nu^{-1}$, which is equal to the period of a pulsar. As GWs pass through our space, the measured elapse $\tau(t)$ will vary slightly, and we have

$$\begin{aligned}\frac{\nu_{\mathrm{o}} - \nu(t)}{\nu_{\mathrm{o}}} &\simeq \frac{\tau(t) - \tau_{\mathrm{o}}}{\tau_{\mathrm{o}}} \text{ (delay of photon transit time)} \\ &\simeq -\frac{\mathrm{i}}{kL}\left(h_+\mathcal{F}_+ + \mathrm{i}h_\times\mathcal{F}_\times\right)\left\{1 - \exp\left[\mathrm{i}kL\left(1 - \cos\gamma_1\right)\right]\right\}\exp\left[-\mathrm{i}\left(kL + \omega_{\mathrm{g}}t_0\right)\right].\end{aligned}$$

For the cumulative variation, we define a ***residual*** [***Detweiler, ApJ (1979)***]:

$$\begin{aligned}\mathfrak{r}(t) &\equiv \int_0^t \frac{\nu_{\mathrm{o}} - \nu(t')}{\nu_{\mathrm{o}}}dt' \simeq \int_0^t \frac{\tau(t') - \tau_{\mathrm{o}}}{\tau_{\mathrm{o}}}dt' \\ &\sim \frac{h_+\mathcal{G}_+(f) + \mathrm{i}h_\times\mathcal{G}_\times(f)}{f}\exp\left(-2\mathrm{i}\pi f t\right),\end{aligned}$$

$$\begin{aligned}\mathcal{G}_+(f) &\equiv \frac{\mathcal{F}_+\exp\left(-\mathrm{i}kL\right)\left\{1 - \exp\left[2\mathrm{i}\pi f\tau_{\mathrm{o}}\left(1 - \cos\gamma_1\right)\right]\right\}}{4\pi^2 f\tau_{\mathrm{o}}}, \\ \mathcal{G}_\times(f) &\equiv \frac{\mathcal{F}_\times\exp\left(-\mathrm{i}kL\right)\left\{1 - \exp\left[2\mathrm{i}\pi f\tau_{\mathrm{o}}\left(1 - \cos\gamma_1\right)\right]\right\}}{4\pi^2 f\tau_{\mathrm{o}}}.\end{aligned}$$

Fourier transformation of $\mathfrak{r}(t)$ yields

$$\tilde{\mathfrak{r}}(f) = \frac{\tilde{h}_+(f)\mathcal{G}_+(f) + \mathrm{i}\tilde{h}_\times(f)\mathcal{G}_\times(f)}{f}.$$

The relation holds

$$\begin{aligned} \left\langle \mathfrak{r}^2(t) \right\rangle_{\text{time}} &\sim f^2 \tilde{\mathfrak{r}}(f)\, \tilde{\mathfrak{r}}^*(f) \\ &\simeq \left|\tilde{h}_+(f)\right|^2 |\mathcal{G}_+(f)|^2 + \left|\tilde{h}_\times(f)\right|^2 |\mathcal{G}_\times(f)|^2. \end{aligned}$$

The detector response function can be computed by taking a *sky average*:

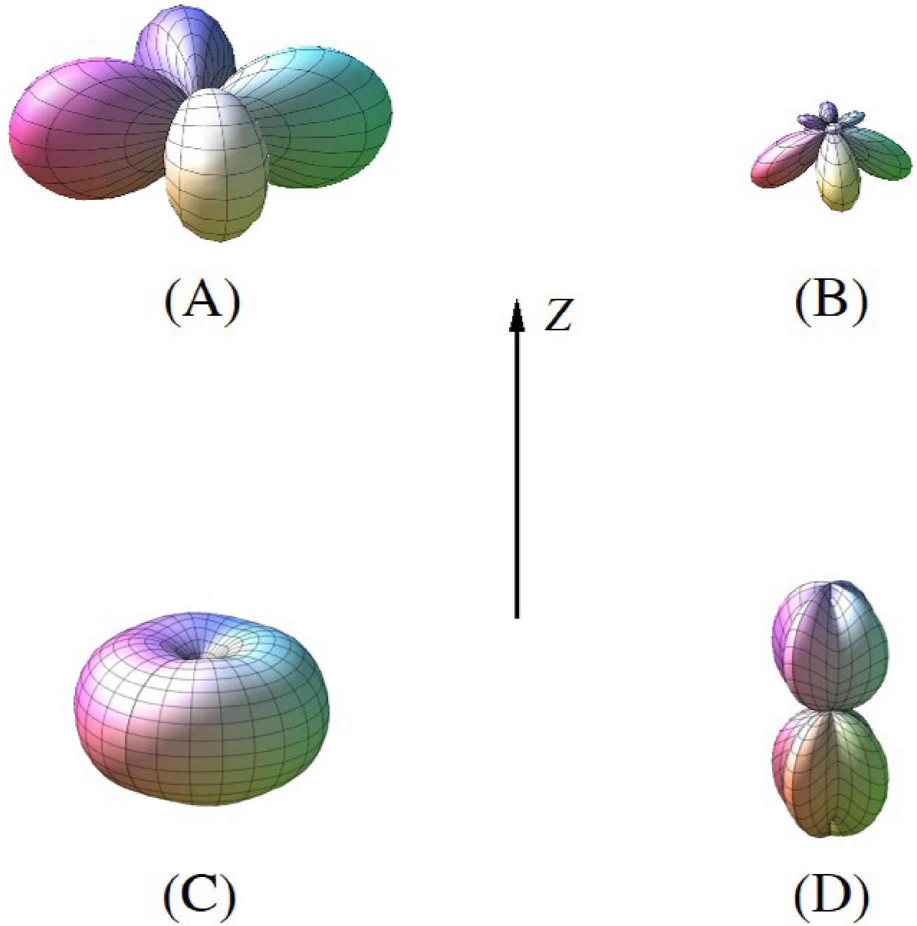
$$\begin{aligned} \mathcal{R}(f) &\equiv \frac{1}{4\pi^2} \int_0^{\pi} d\psi \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta \left[\mathcal{G}_+(f)\, \mathcal{G}_+^*(f) + \mathcal{G}_\times(f)\, \mathcal{G}_\times^*(f)\right] \\ &= \begin{cases} \dfrac{32\pi^3 f^3 \tau_\mathrm{o}^3 - 12\pi f \tau_\mathrm{o} + 3\sin(4\pi f \tau_\mathrm{o})}{768\pi^7 f^5 \tau_\mathrm{o}^5} & \textbf{for } \mathbf{K} = (0, 0, -K), \\ \dfrac{29 + 150\cos^2\theta_\star - 115\cos^4\theta_\star}{1920\pi^2} + \mathcal{O}\left(f^2\tau_\mathrm{o}^2\right) & \textbf{for } \mathbf{K} = (K_x, K_y, K_z) \textbf{ and } f\tau_\mathrm{o} \ll 1. \end{cases} \end{aligned}$$

However, one can infer

$$\tilde{\mathfrak{r}}(f)\, \tilde{\mathfrak{r}}^*(f) \sim f^{-2} \left|\tilde{h}(f)\right|^2 \mathcal{R}(f).$$

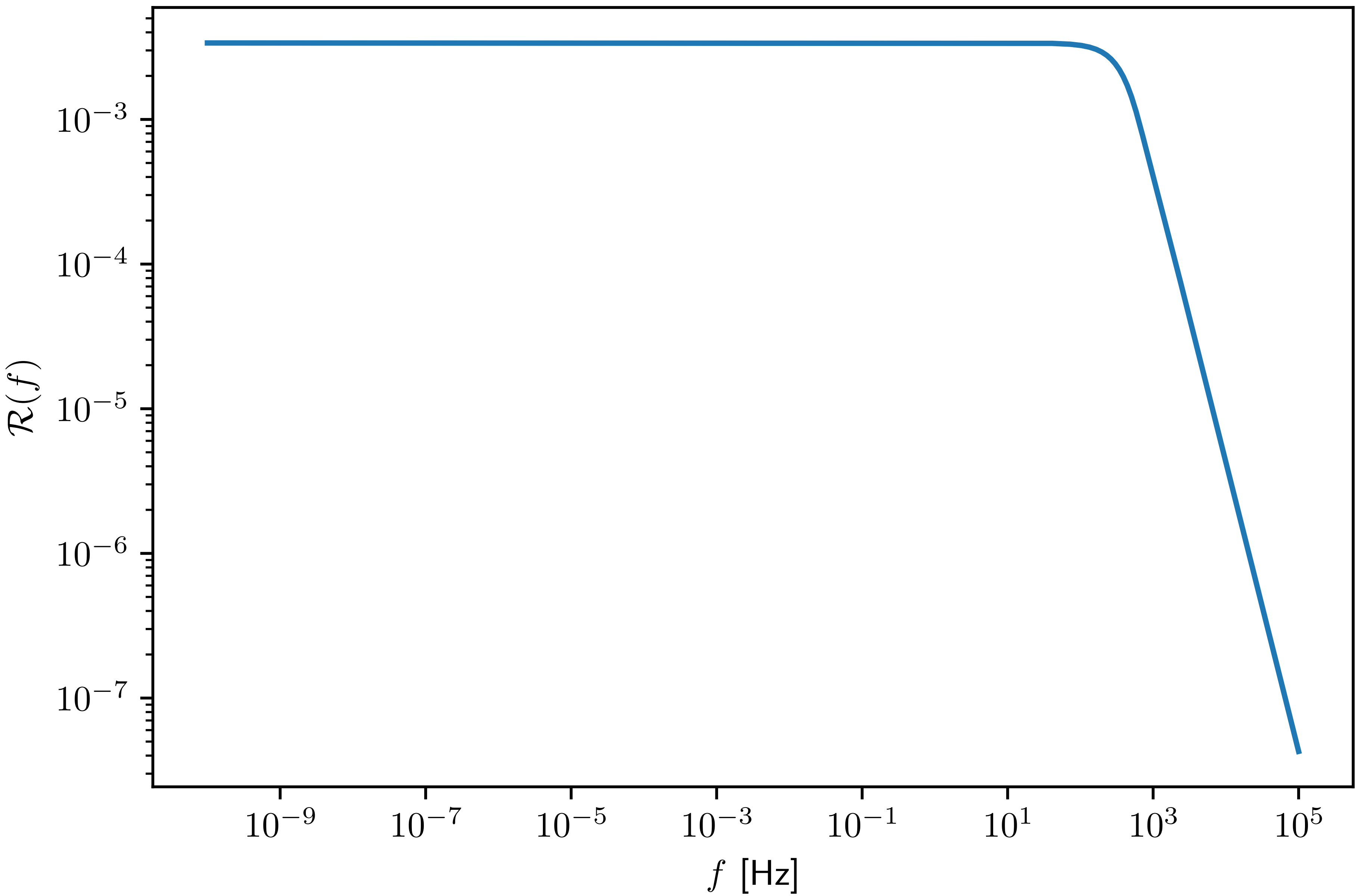
Then, the sensitivity can be determined from

$$h(f) \equiv f\tilde{h}(f) \sim \sqrt{\frac{f^2 \left\langle \mathfrak{r}^2(t) \right\rangle_{\text{time}}}{\mathcal{R}(f)}}.$$

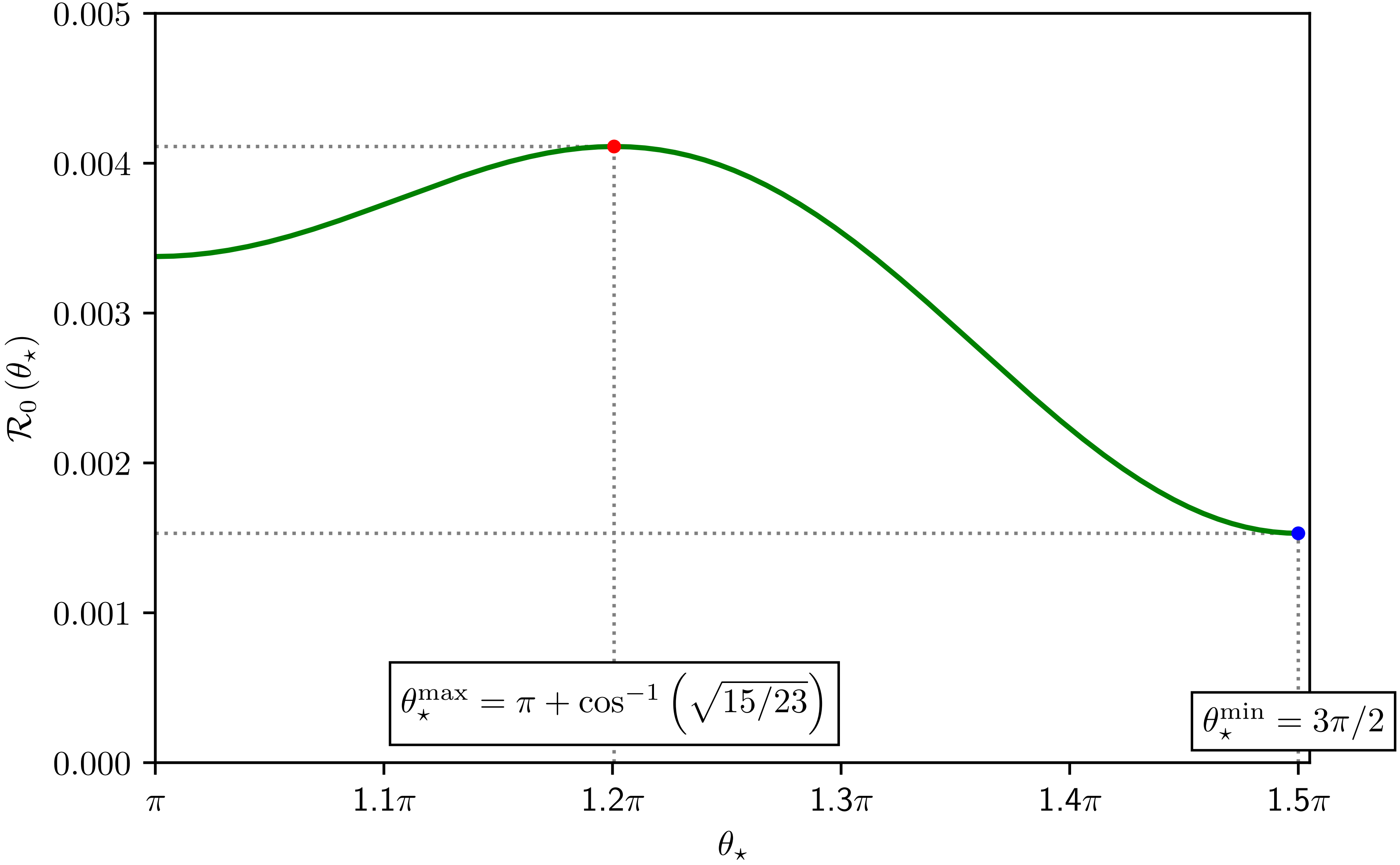


$$X = \sin\theta\cos\psi, \qquad Y = \sin\theta\sin\psi, \qquad Z = \cos\theta$$

Plots of antenna patterns for the detector responses: for light from a millisecond pulsar with $\tau_{\rm o} \sim 10^{-3}$ s,
(A) $|\mathcal{G}_{+,\times}|$ at $f \ll 1\,$Hz, (B) $|\mathcal{G}_{+,\times}|$ at $f = 1000\,$Hz for $\mathbf{K} = (0, 0, -K)$;
(C) $|\mathcal{G}_{+,\times}|$ at $\theta_\star = \pi + \cos^{-1}\left(\sqrt{15/23}\right)$, (D) $|\mathcal{G}_{+,\times}|$ at $\theta_\star = 3\pi/2$ for $\mathbf{K} = (K_x, K_y, K_z)$ in the regime $f \ll 1\,$Hz.



Plot of $\mathcal{R}(f)$ for light with $\mathbf{K} = (0, 0, -K)$ from a millisecond pulsar with $\tau_o \sim 10^{-3}$ s.



Plot of $\mathcal{R}_0(\theta_\star) = \left(29 + 150\cos^2\theta_\star - 115\cos^4\theta_\star\right) / \left(1920\pi^2\right)$ for light with $\mathbf{K} = (K_x, K_y, K_z)$ from a millisecond pulsar with $\tau_o \sim 10^{-3}$ s; having the maximum at $\theta_\star = \pi + \cos^{-1}\left(\sqrt{15/23}\right) \approx 216^\circ$ and the minimum at $\theta_\star = 3\pi/2$.

(2) Sensitivity curves for PTA

The sensitivity of our PTA can be determined from

$$h(f) \sim \sqrt{\frac{f^2 \langle \mathfrak{r}^2(t) \rangle_{\text{time}}}{\mathcal{R}(f)}}.$$

Now, we can estimate the r.m.s. of ***residual***:

$$\sqrt{\langle \mathfrak{r}^2(t) \rangle} \simeq \sqrt{\left\langle \left[\frac{1}{2c\tau_{\text{o}}} \int_0^t \int_{c\tau_{\text{o}}}^0 h_{zz}(t, 0, 0, z)\, dz dt \right]^2 \right\rangle}$$
$$\sim \omega_{\text{g}}^{-1} h_{\text{max}}.$$

For example, consider a periodic source of GWs: two supermassive black holes of mass M in a circular orbit of radius R_{o} about one another, the luminosity distance of which is r. Then we have [***Detweiler, ApJ (1979)***]

$$h_{\text{max}} \sim 5 \times 10^{-14} \left(\frac{200M}{R_{\text{o}}} \right) \left(\frac{M}{10^{10} M_{\odot}} \right) \left(\frac{10^{10}\ \text{ly}}{r} \right)$$

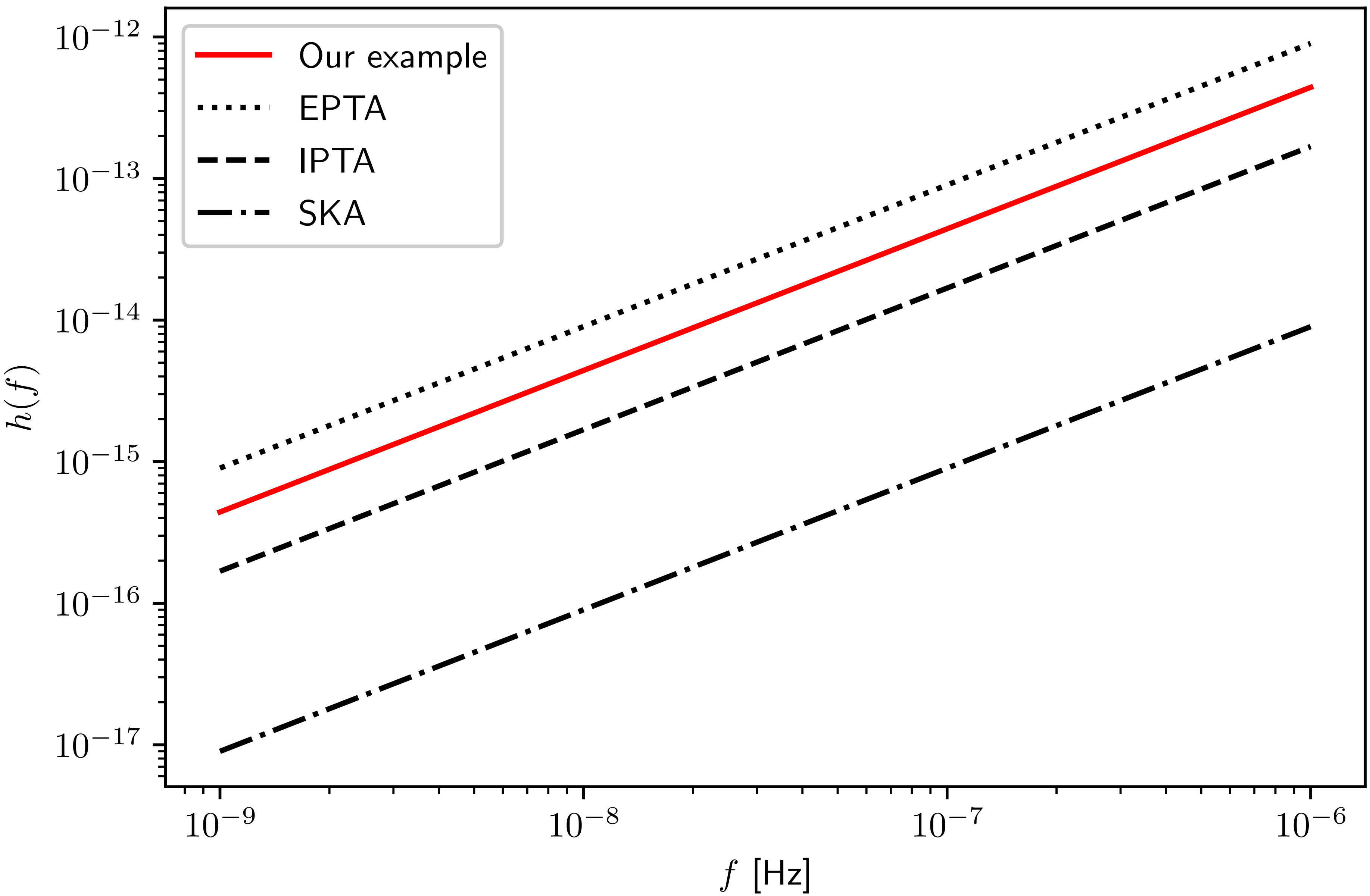
and

$$\omega_{\text{g}} \sim 2 \times 10^{-8}\ \text{s}^{-1} \left(\frac{200M}{R_{\text{o}}} \right)^{3/2} \left(\frac{10^{10} M_{\odot}}{M} \right).$$

Therefore,

$$\sqrt{\langle \mathfrak{r}^2(t) \rangle} \sim 2 \times 10^{-6} \left(\frac{R_{\text{o}}}{200M} \right)^{1/2} \left(\frac{M}{10^{10} M_{\odot}} \right)^2 \left(\frac{10^{10}\ \text{ly}}{r} \right).$$

With $\mathcal{R}(f)$ and $\langle \mathfrak{r}^2(t) \rangle$ determined, the sensitivity curves for our PTA are obtained: *e.g.* from the GW source with $M \sim 10^9 M_\odot$, $R_\mathrm{o} \sim 2 \times 10^{11} M_\odot$ and $r \sim 10^{10}$ ly.



Plot of $h(f)$ for $\tau_\mathrm{o} \sim 10\mathrm{ms}$ in comparison with EPTA, IPTA, SKA curves [*Moore et al., CQG (2015)*].

4. Conclusions and discussion

- A perturbation of light due to GWs is physically *equivalent* to a delay of the photon transit time: Maxwell's equations vs. null geodesic equation

$$\frac{\delta T_{[h]}}{T} \simeq \mathcal{N} \left[\left. \frac{\delta E^i_{[h]}}{E^i_{\rm o}} \right|_{z=0} - \left. \frac{\delta E^i_{[h]}}{E^i_{\rm o}} \right|_{z=L} \right] ; \quad \mathcal{N} = (\mathrm{i}\omega_{\rm e} T)^{-1} = (\mathrm{i}KL)^{-1} .$$

- To determine the effects of GWs via a PTA, we may consider the variation of the elapse $\tau(t) = \nu^{-1}(t)$, instead of the variation of the frequency $\nu(t)$. Then it will be equivalent to the delay of photon transit time:

$$\frac{\nu_{\rm o} - \nu(t)}{\nu_{\rm o}} \simeq \frac{\tau(t) - \tau_{\rm o}}{\tau_{\rm o}}.$$

- We have determined the response function and the residual to construct a sensitivity curve for a PTA. Our results are in good agreement with the literature.

- Our analysis can be extended to more complex arrays for GW detection than a PTA: e.g., interferometers such as LIGO and LISA, which require a description of light rays in more complicated configurations. We leave further analysis to a follow-up study.