Second Order Measurements in Interferometric Gravitational Wave Detectors

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2021.07.09

The 8th KAGRA International Workshop

Introduction

Second Order Measurement in Interferometers

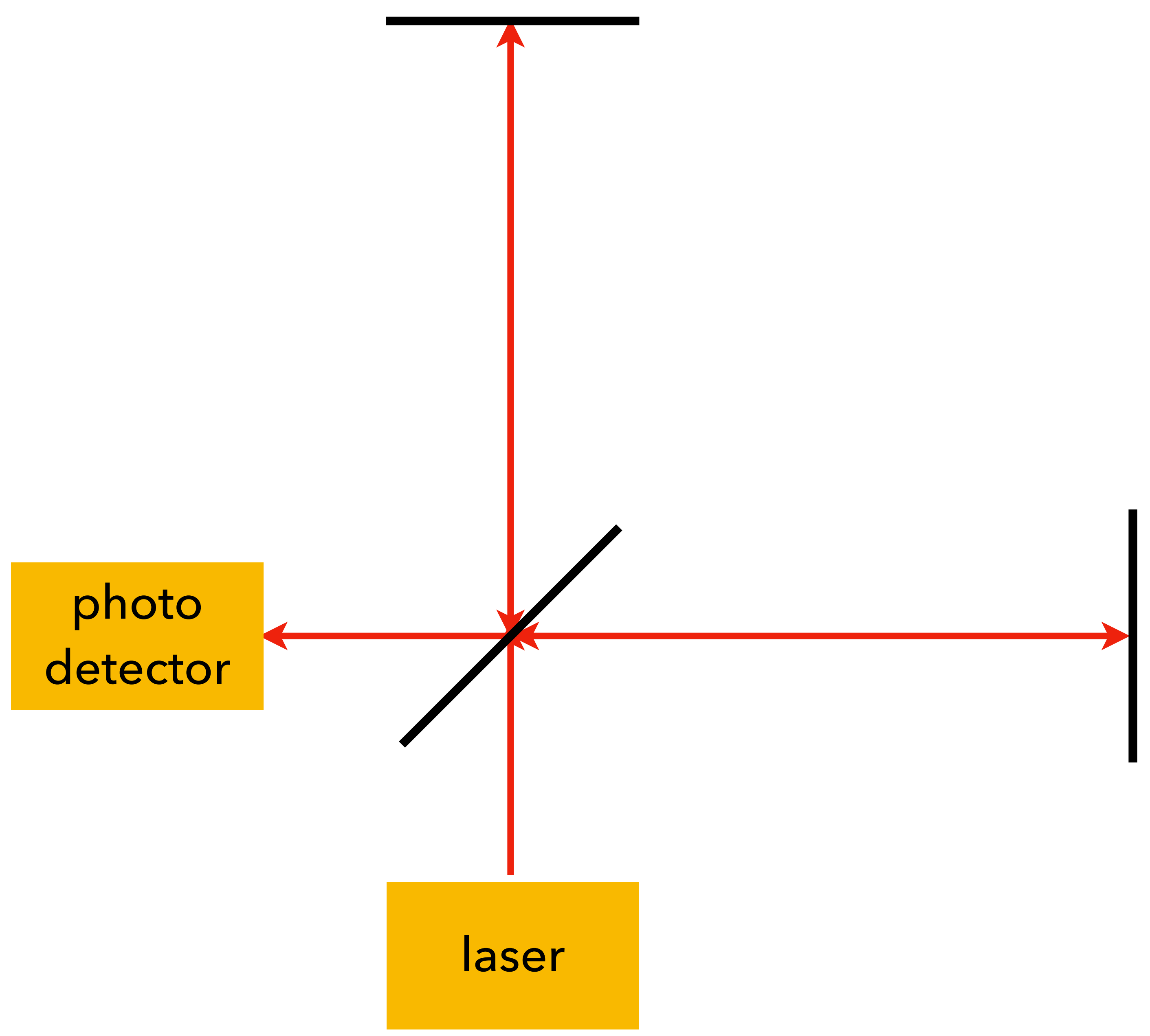
Background dark port

Second order measurement

$$\frac{I}{\mathcal{I}} = \mathcal{O}\left(\alpha^2\right) + \mathcal{O}\left(\alpha h\right) + \mathcal{O}\left(\alpha^2 h\right) + \cdots$$

First order measurement

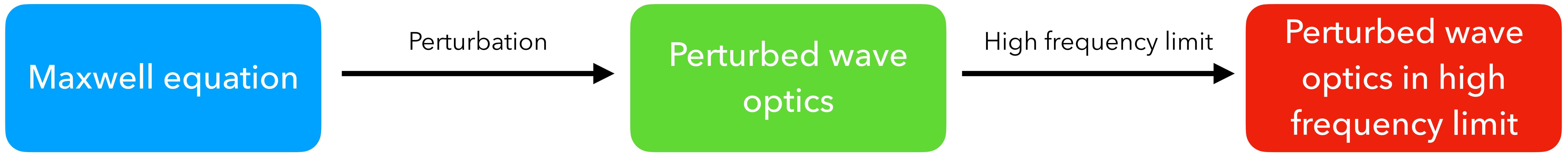
- I: dark port intensity at photo detector
- $\mathcal{I}$: source power
- α: phase difference between two waves
- h: gravitational wave strength



Two Approaches for Interferometer Analysis

- Perturbed geometrical optics

- Perturbed wave optics in high frequency limit



Why Perturbed Wave Optics?

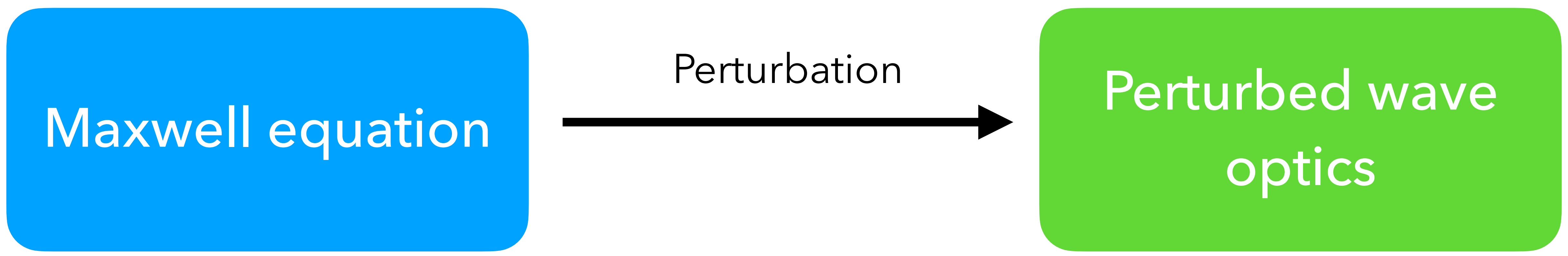
	Perturbed geometrical optics	Perturbed wave optics in high frequency limit
Pros	familiar and simple if we only consider phase.	All terms regarding phase and amplitude can be considered. Natural extension to wave optics is possible.
Cons	not easy to consider amplitude.	extremely complex.

- However, we believe that two approaches are identical. (verification is required)

Perturbation of EMW

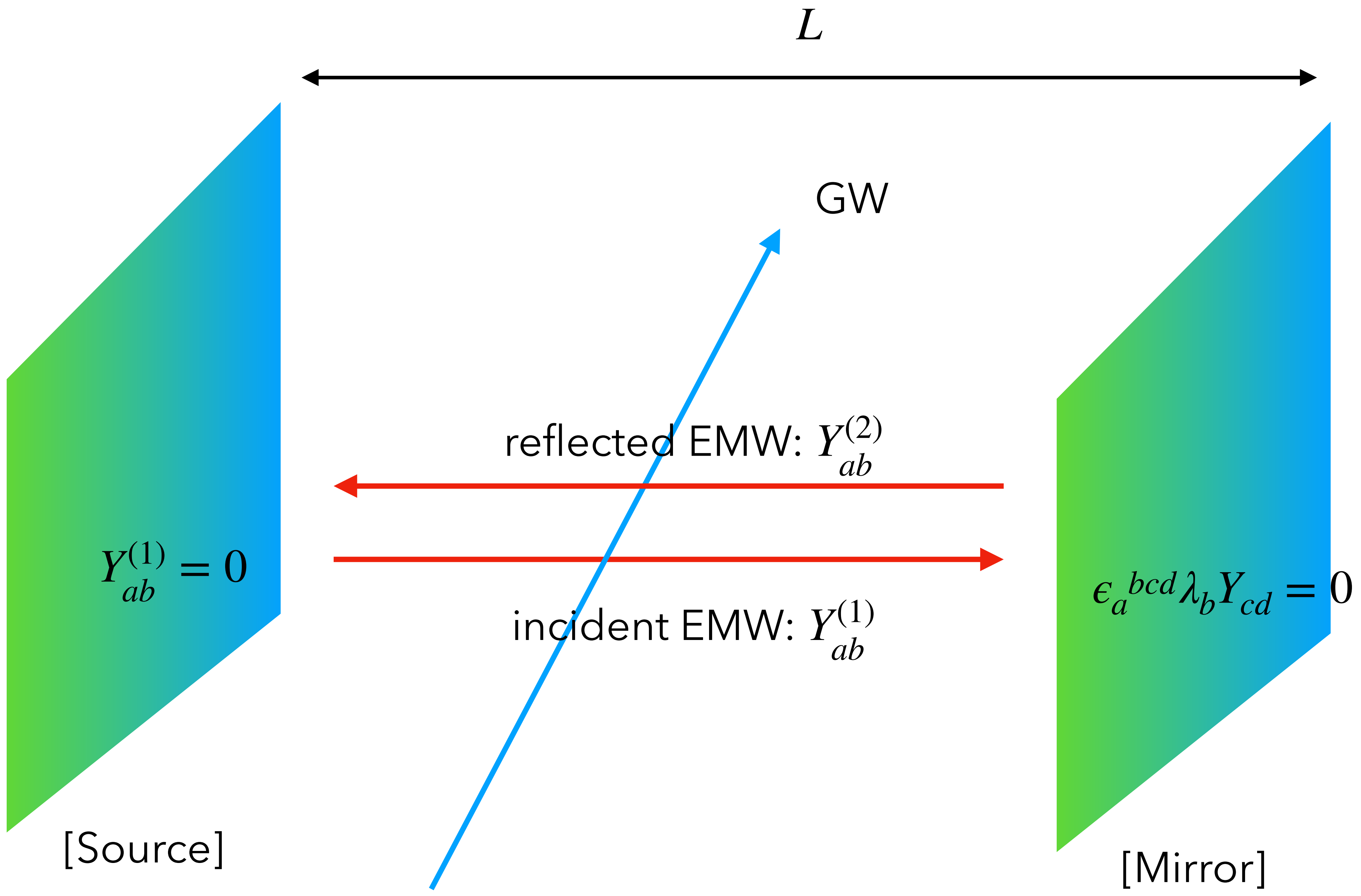
Perturbation of EMW

- Perturbed potential
  - ${}^{\epsilon}A_a = \epsilon A_a + \epsilon^2 X_a + O\left(\epsilon^3\right)$
- Perturbed Maxwell equation
  - $\nabla^b \nabla_b X_a = 2 \nabla_b A_c C^{c}{}_{a}{}^{b} + h^{bc} \nabla_b \nabla_c A_a$
- General solution
  - $X_a = X_a^{\mathrm{p}} + X_a^{\mathrm{h}}$
  - Particular solution is determined by GW and EMW.
  - Homogeneous solution is determined by given boundary condition



Boundary Condition with Source and Mirror

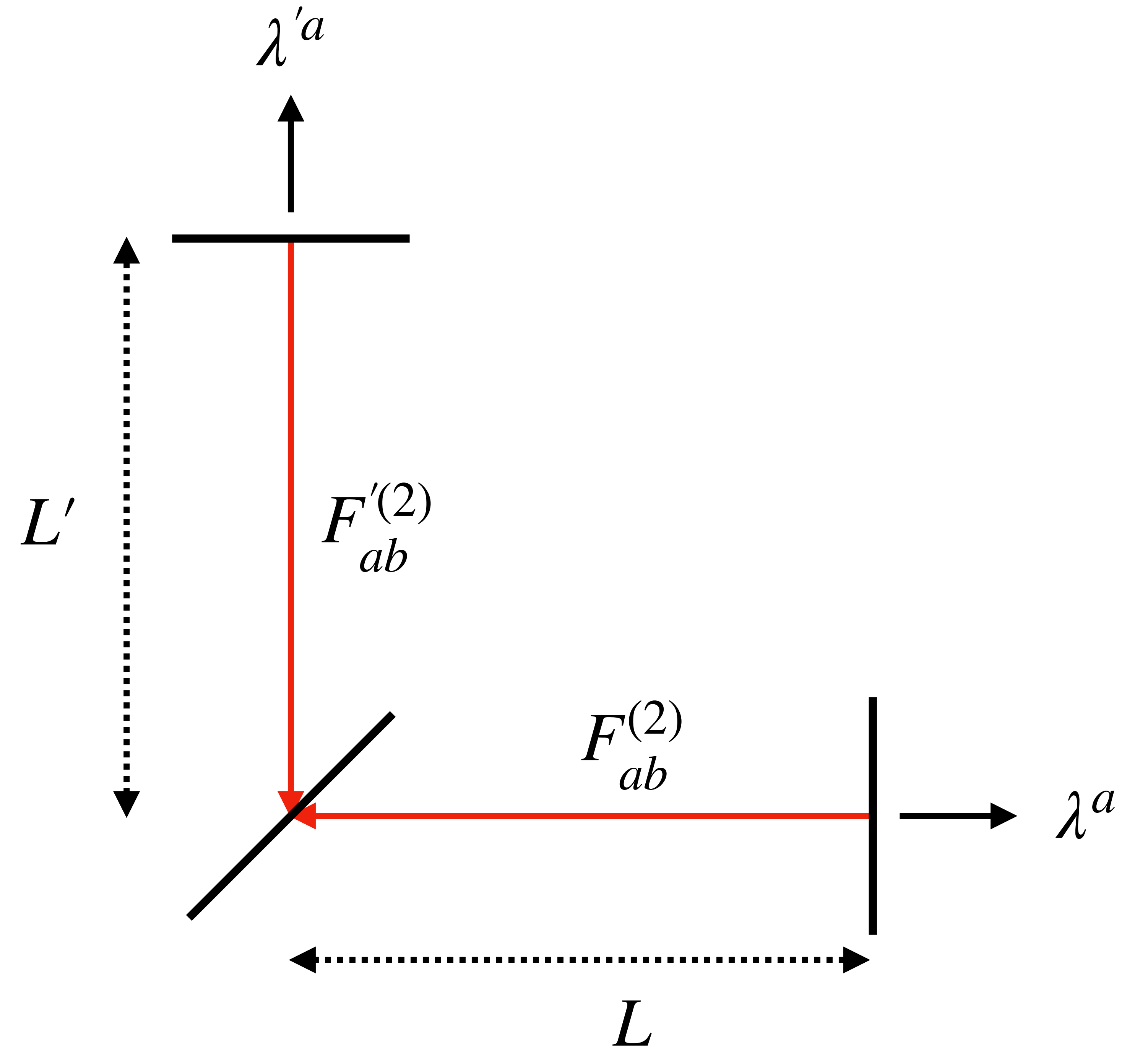
- Perturbed EM field
  - ${}^{\epsilon}F_{ab} = \epsilon F_{ab} + \epsilon^2 Y_{ab} + O\left(\epsilon^3\right)$
  - $Y_{ab} = 2\nabla_{[a}X_{b]}$
- Perturbations for two waves
  - $Y^{(1)}_{ab}$: incident wave perturbation
  - $Y^{(2)}_{ab}$: reflected wave perturbation
  - $Y_{ab} = Y^{(1)}_{ab} + Y^{(2)}_{ab}$
- Boundary condition
  - $Y^{(1)}_{ab} = 0$ at source ($\vec{x} = 0$)
  - $\epsilon_a{}^{bcd}\lambda_b Y_{cd} = 0$ at mirror ($\vec{x} = L\lambda$)



Interferometer

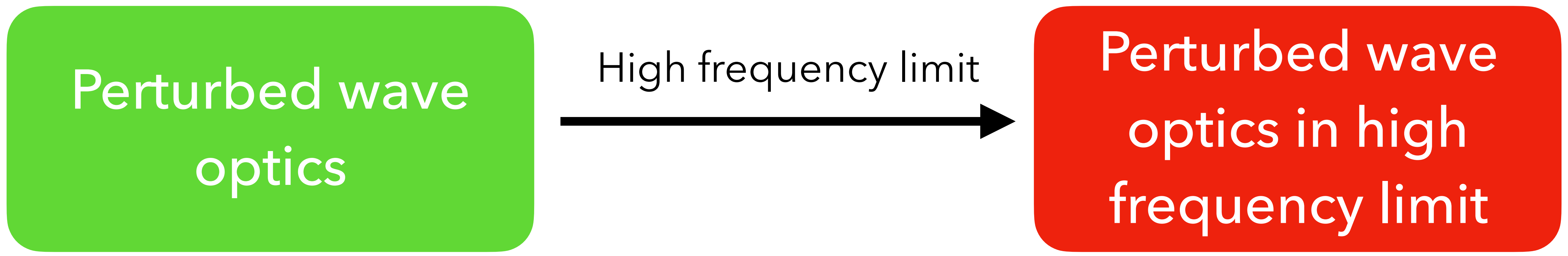
Background

- Ansatz
  - $\tilde{E}_a = \tilde{E}'_a = \mathscr{E} p_a e^{\mathrm{i}\delta_e}$ where $p^a \equiv \epsilon^a{}_{bc}\lambda^b\lambda'^c$
- Received EMW
  - $F^{\mathrm{r}}_{ab}(t,0) = F^{(2)}_{ab}(t,0) + \mathscr{R}F'^{(2)}_{ab}(t,0)$
  - $E^{\mathrm{r}}_a = F^{\mathrm{r}}_{ab}n^b \qquad B^{\mathrm{r}}_a = \frac{1}{2}\epsilon^{bc}{}_a F^{\mathrm{r}}_{bc}$
- Intensity at dark port
  - $I^{\mathrm{r}} = \frac{1}{4\pi}\epsilon^{ab}{}_c E^{\mathrm{r}}_a B^{\mathrm{r}}_b \lambda^c = 4\mathscr{I}\left(1 + \cos\left(2\mathrm{i}\omega_{\mathrm{e}}(L - L')\right)\right)$
  - $2\mathrm{i}\omega_{\mathrm{e}}(L - L') = \pi + \alpha$ where $|\alpha| \ll 1$
  - $\frac{I^{\mathrm{r}}}{\mathscr{I}} \simeq 2\alpha^2$



Perturbation of Dark Port Intensity

- High frequency limit
    - We only take leading order terms
    - proportional to $O\left(\omega_{\mathrm{e}}/\omega_{\mathrm{g}}\right)$.



- Incomplete interim results

$$\frac{\delta I^{\mathrm{r}}}{\mathcal{I}} = 4\Re\left[\frac{\omega_{\mathrm{e}}}{-\mathrm{i}\omega_{\mathrm{g}}}\tilde{h}_{ab}e^{-\mathrm{i}\omega_{\mathrm{g}}t}\left\{\frac{\lambda^a\lambda^b}{1+\cos\theta}\left(1-e^{\mathrm{i}\Delta^{\mathrm{m}}}\right)\alpha+\frac{\lambda^a\lambda^b}{1-\cos\theta}\left(\sin\left(\omega_{\mathrm{e}}\left(L+L'\right)\right)+e^{\mathrm{i}\Delta^{\mathrm{m}}}\right)\alpha\right.\right.$$

- $$\left.\left.-\frac{\lambda^a\lambda^b}{1+\cos\theta}\left(1-e^{\mathrm{i}\Delta^{\mathrm{m}}}\right)\alpha+\frac{\lambda^a\lambda^b}{1-\cos\theta}\left(\sin\left(\omega_{\mathrm{e}}\left(L+L'\right)\right)-e^{\mathrm{i}\Delta^{\mathrm{m}}}\right)\alpha+O\left(\alpha^2\right)\right\}\right]$$

Summary

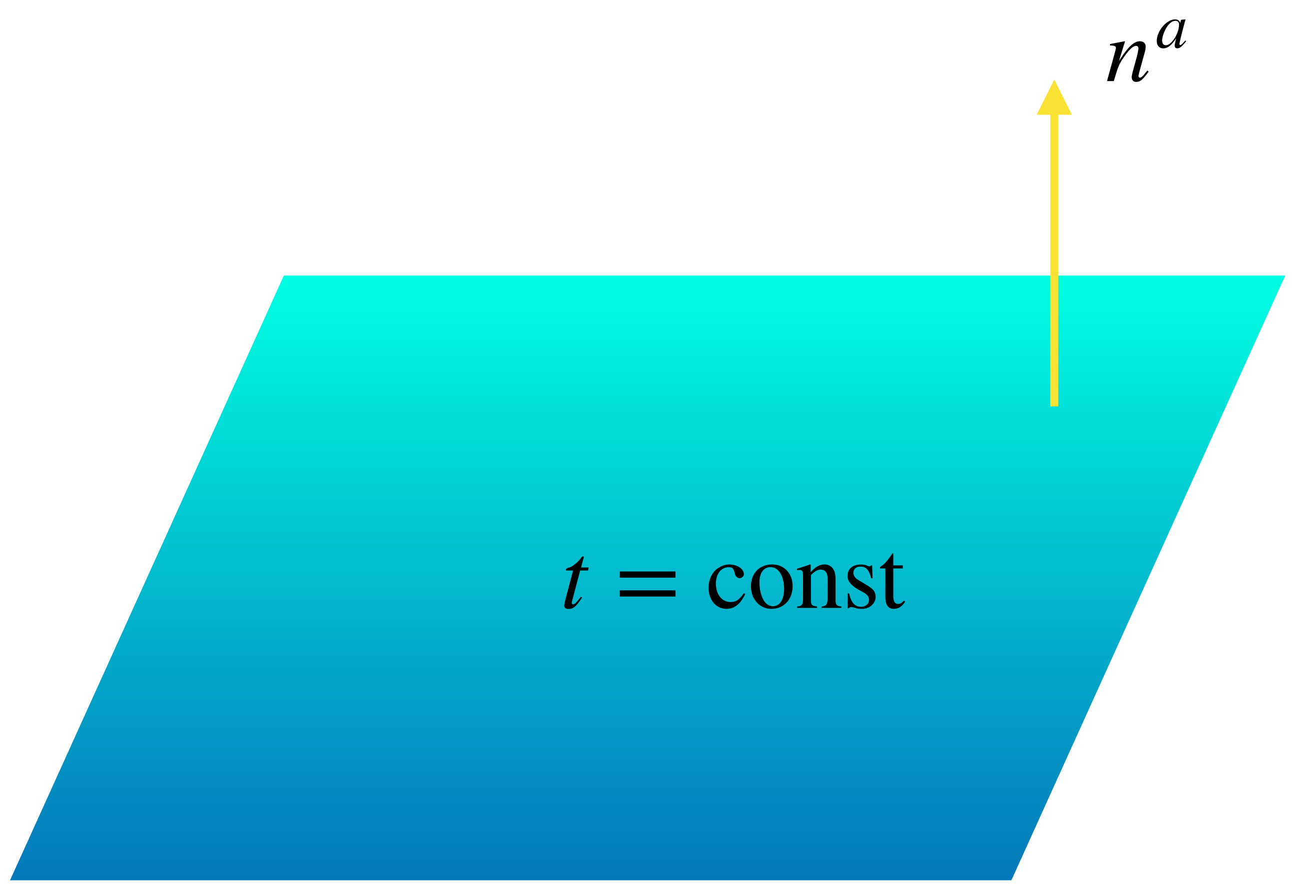
- We provided analysis of perturbed EMW in wave optics.
- Using boundary conditions for source and mirror, we can get general solution of perturbed EMW.
- We provided analysis of interferometer system using perturbed wave optics.
- We are looking analytic form for the intensity of superposed waves from two arm.
- It is expected that the analytic results extend our options in future detector design.

Supplementary Slides

Gravitational Waves

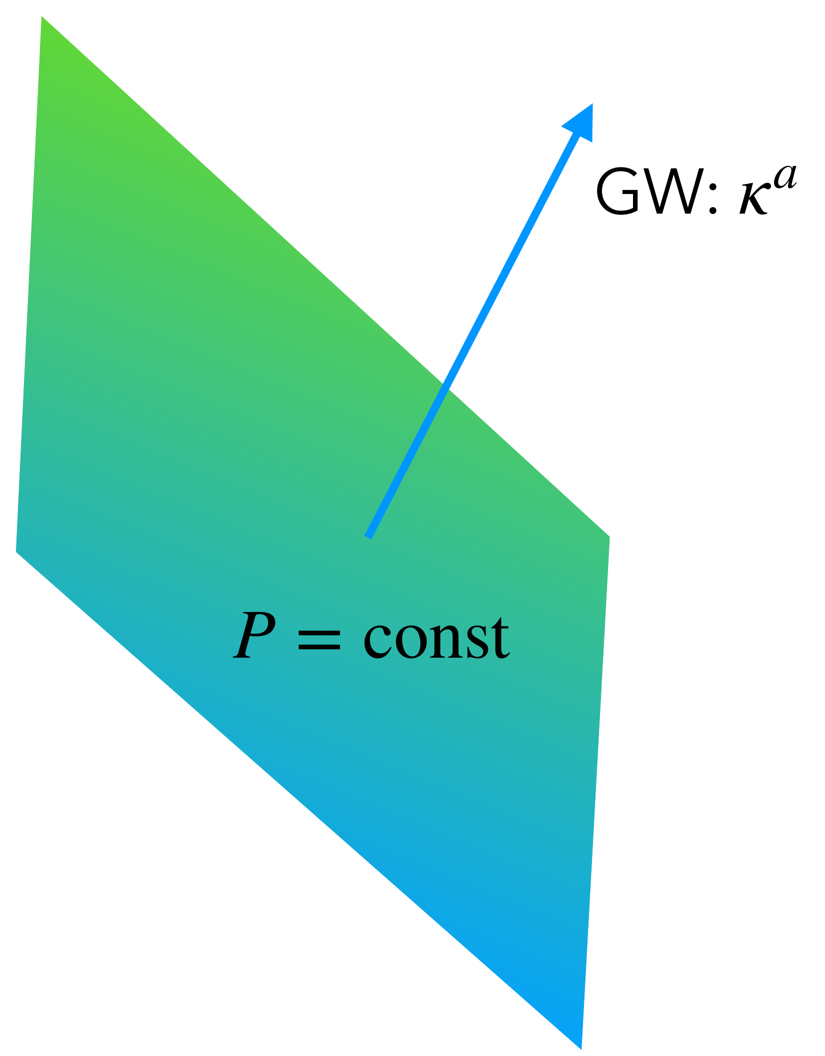
Minkowski Background

- g_{ab}: Minkowski background metric
- ∇: Levi-Civita connection associated with g_{ab}
- t: a global inertial time coordinate
- Normal vector of t (4-velocity of Eulerian observer)
  - $n^a \equiv -g^{ab}\nabla_b t$



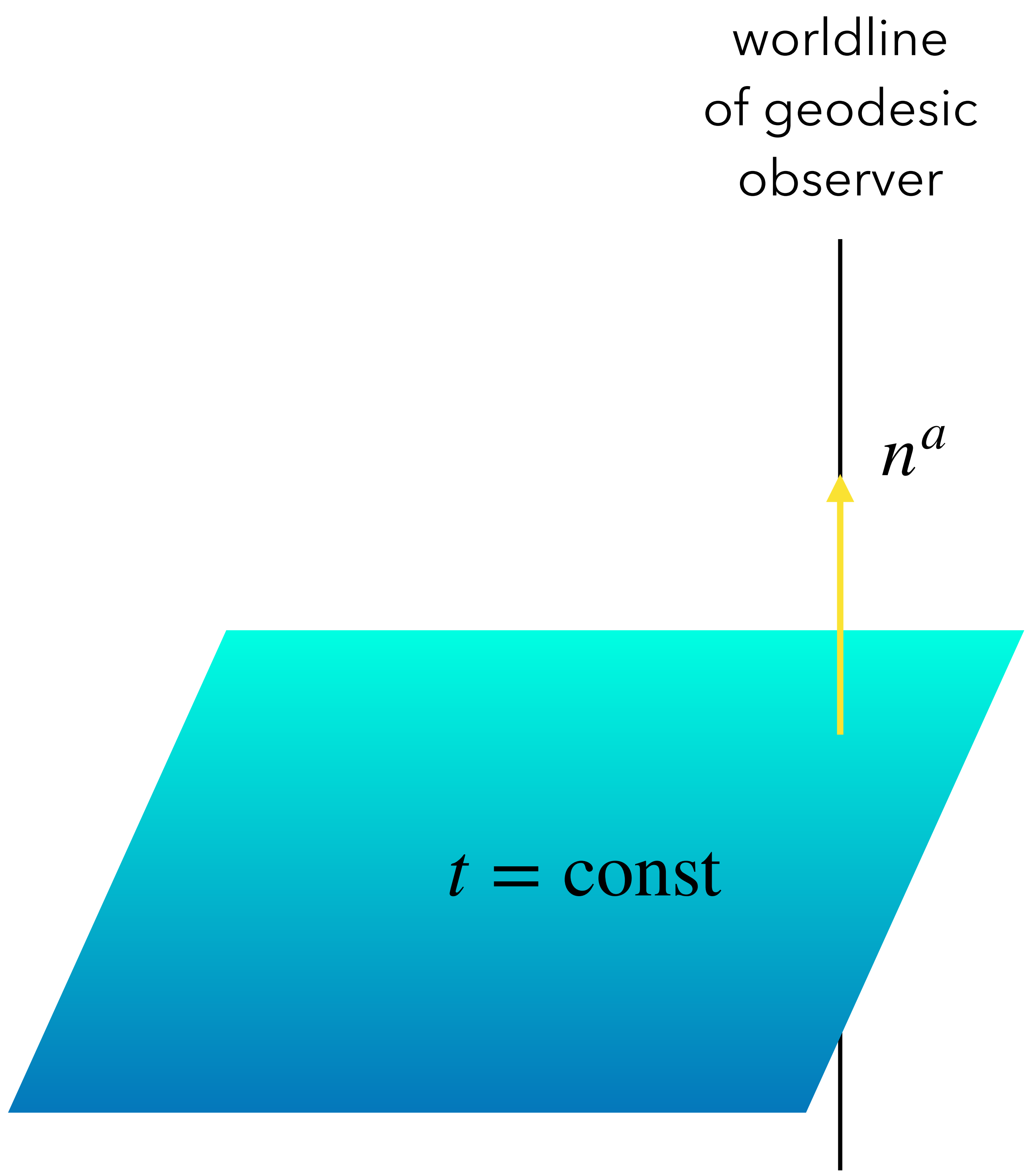
Gravitational Waves (GW)

- Perturbed metric
  - ${}^{\epsilon}g_{ab} = g_{ab} + \epsilon h_{ab} + O\left(\epsilon^2\right)$
- Perturbed vacuum Einstein equation
  - $0 = \nabla^c \nabla_c h_{ab}$
  - $0 = \nabla^b h_{ab}$
- Monochromatic plane GW
  - $h_{ab}\left(t, \vec{x}\right) = \tilde{h}_{ab} e^{\mathrm{i}P\left(t, \vec{x}\right)}$
  - $\tilde{h}_{ab}$: amplitude of GW
  - $P\left(t, \vec{x}\right) \equiv \omega_{\mathrm{g}}\left(-t + \kappa \cdot \vec{x}\right)$: phase of GW
  - ω_{g}: frequency of GW
  - κ^a: spatial unit vector for GW propagation
  - $k_a \equiv \nabla_a P = \omega_{\mathrm{g}}\left(n_a + \kappa_a\right)$: wave vector of GW
- Properties
  - $0 = g_{ab} k^a k^b$: null condition
  - $0 = k^b \tilde{h}_{ab}$: Lorentz gauge
  - $0 = n^b \tilde{h}_{ab}$: radiation gauge



Geodesic Observers

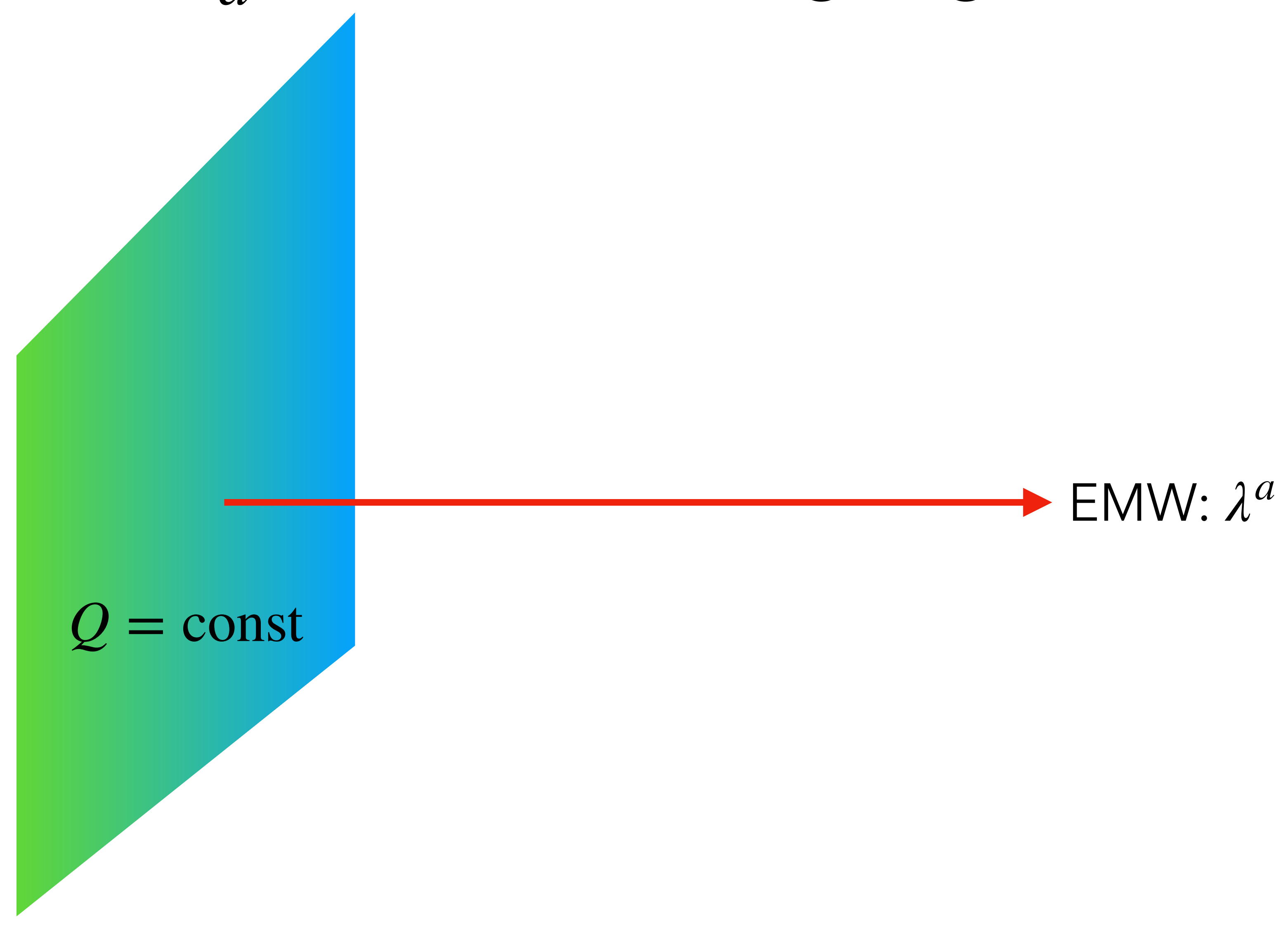
- Perturbed 4-velocity of geodesic observers
  - ${}^{\epsilon}n^a = n^a + \epsilon \delta n^a + O\left(\epsilon^2\right)$
- Perturbed geodesic equation
  - $n^b \nabla_b \delta n^a = 0$
- We can choose geodesic observers with $\delta n^a = 0$.



Electromagnetic Waves

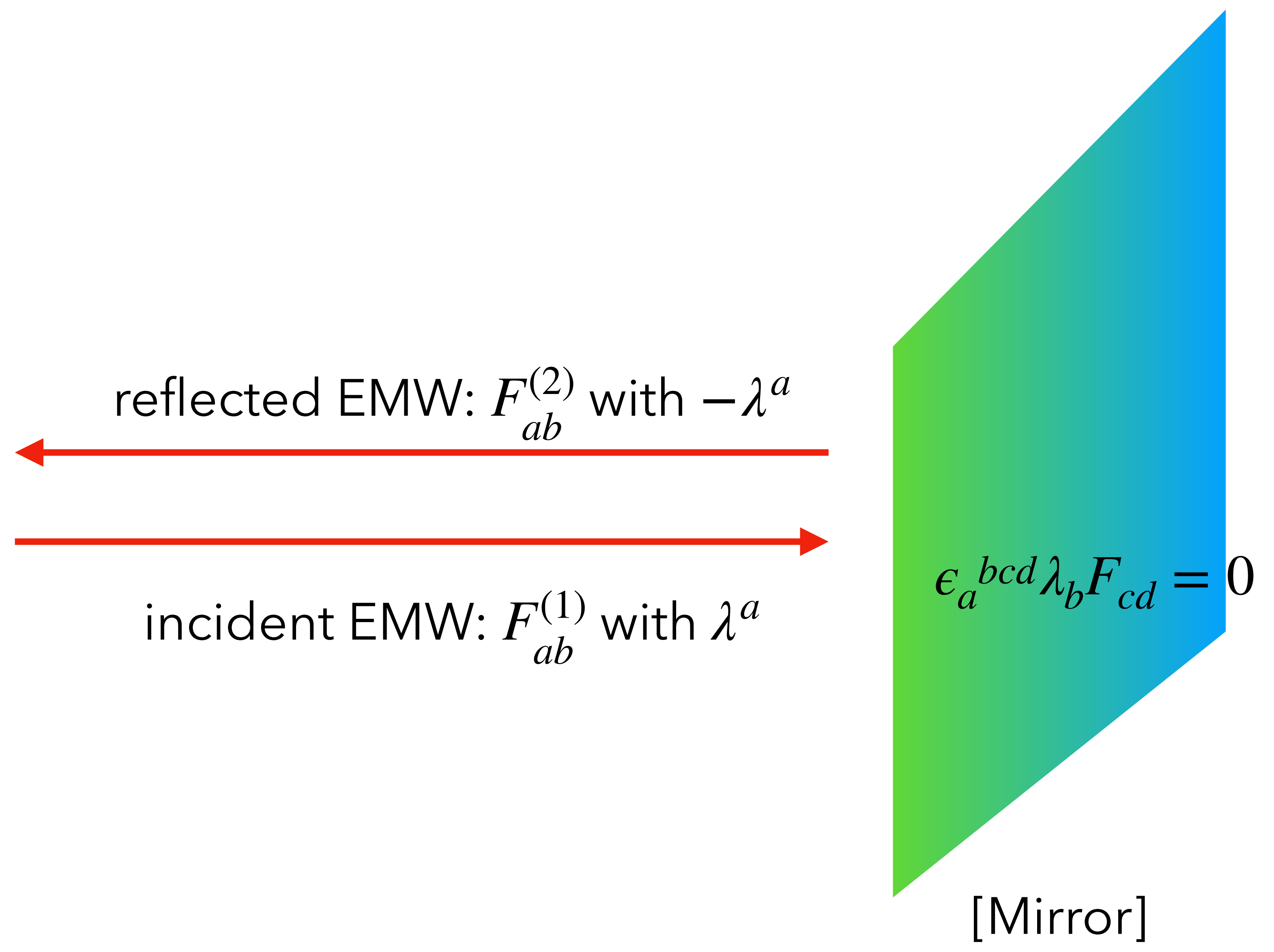
Electromagnetic Wave (EMW)

- Vacuum Maxwell equation
  - $0 = \nabla^b \nabla_b A_a$
  - $0 = \nabla^a A_a$
- Monochromatic plane EMW
  - $A_a\left(t, \vec{x}\right) = \tilde{A}_a e^{\mathrm{i}Q\left(t, \vec{x}\right)}$
  - $\tilde{A}_a$: amplitude of EMW
  - $Q\left(t, \vec{x}\right) \equiv \omega_{\mathrm{e}}\left(-t + \lambda \cdot \vec{x}\right)$: phase of EMW
  - ω_{e}: frequency of EMW
  - λ^a: spatial unit vector for EMW propagation
  - $l_a \equiv \nabla_a Q = \omega_{\mathrm{e}}\left(n_a + \lambda_a\right)$: wave vector of EMW
- Properties
  - $0 = g_{ab} l^a l^b$: null condition
  - $0 = l^a \tilde{A}_a$: Lorentz gauge
  - $0 = n^a \tilde{A}_a$: radiation gauge



Boundary Condition at Mirror

- Two waves
  - $F^{(1)}_{ab}\left(t,\vec{x}\right) = 2\left(n_{[a}+\lambda_{[a}\right)\tilde{E}_{b]}e^{i\omega_e\left(-t+\lambda\cdot\vec{x}\right)}$
  - $F^{(2)}_{ab}\left(t,\vec{x}\right) = -2\left(n_{[a}-\lambda_{[a}\right)\tilde{E}_{b]}e^{i\omega_e\left(-t-\lambda\cdot\vec{x}+2L\right)}$
  - $F_{ab} = F^{(1)}_{ab} + F^{(2)}_{ab}$
- Boundary condition
  - $\epsilon_a{}^{bcd}\lambda_b F_{cd} = 0$ at mirror ($\vec{x} = L\lambda$)



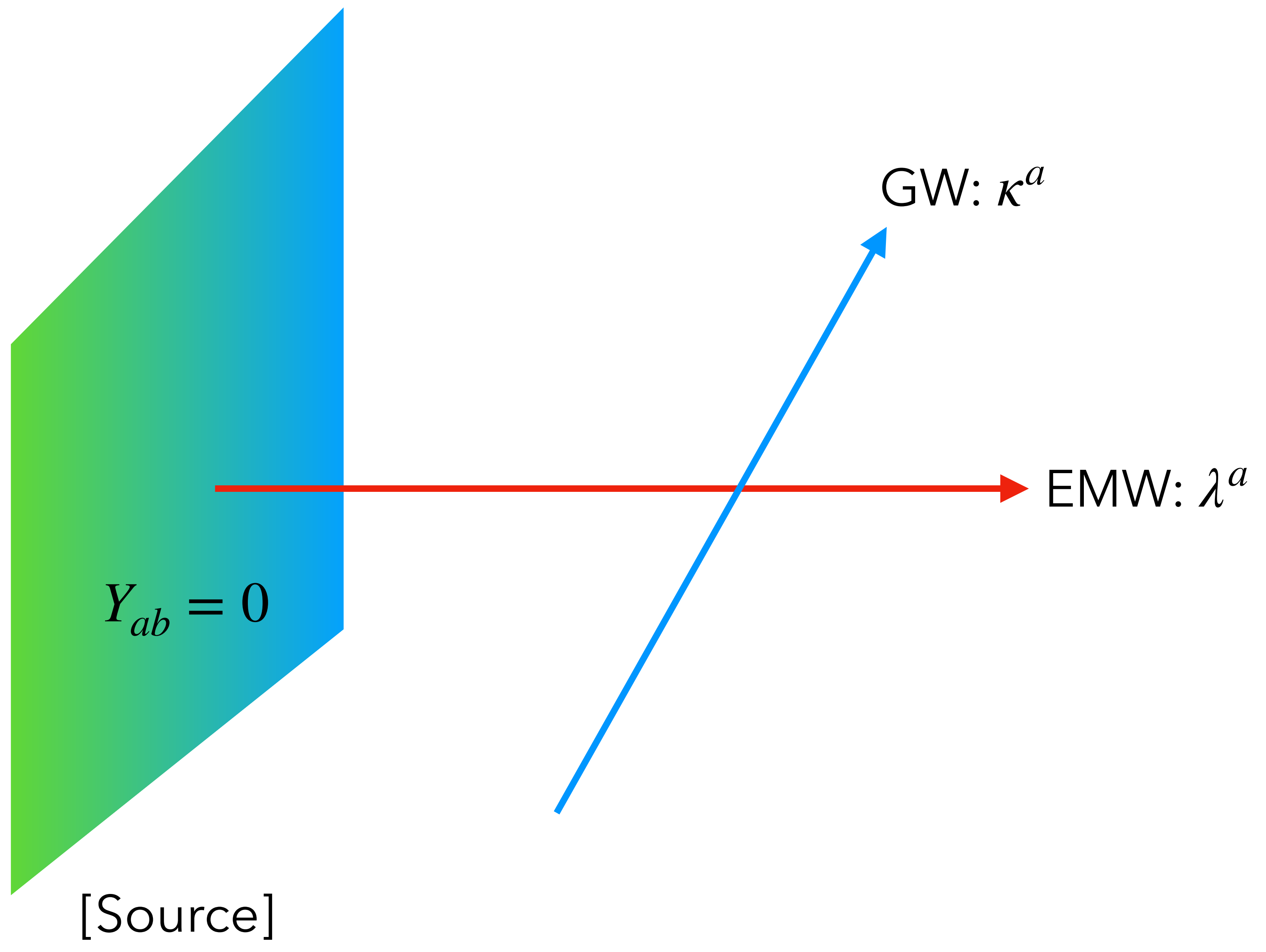
Perturbation of EMW

Perturbation of EMW

- Perturbed potential
  - ${}^{\epsilon}A_a = \epsilon A_a + \epsilon^2 X_a + O\left(\epsilon^3\right)$
- Perturbed Maxwell equation
  - $\nabla^b \nabla_b X_a = 2 \nabla_b A_c C^c{}_a{}^b + h^{bc} \nabla_b \nabla_c A_a$
  - $\nabla^a X_a = h^{ab} \nabla_a A_b$
  - where $C^a{}_{bc} \equiv \frac{1}{2}\left(\nabla_c h^a{}_b + \nabla_b h^a{}_c - \nabla^a h_{bc}\right)$
- General solution
  - $X_a = X_a^{\mathrm{p}} + X_a^{\mathrm{h}}$
- Particular solution
  - $X_a^{\mathrm{p}} = \tilde{X}_a^{\mathrm{p}} e^{\mathrm{i}P} e^{\mathrm{i}Q}$
  - $X_a^{\mathrm{p}} = \frac{1}{2k_a l^a}\left(2\mathrm{i} C^c{}_{ab} l^b A_c - h_{bc} l^b l^c A_a\right)$
- Homogeneous solution
  - $\nabla^b \nabla_b X_a^{\mathrm{h}} = 0$
  - $\nabla^a X_a^{\mathrm{h}} = 0$
  - is determined by given boundary condition
- Form of homogeneous solution
  - $X_a^{\mathrm{h}}\left(t, \vec{x}\right) = \int_{-\infty}^{\infty} \frac{d\omega_{\mathrm{h}}}{2\pi} \int d^2\mu \tilde{X}_a^{\mathrm{h}}\left(\omega_{\mathrm{h}}, \mu\right) e^{iS\left(t, \vec{x}\right)}$
  - $S\left(t, \vec{x}\right) \equiv \omega_{\mathrm{h}}\left(-t + \mu \cdot \vec{x}\right)$
  - $m_a \equiv \nabla_a S = \omega_{\mathrm{h}}\left(n_a + \mu_a\right)$
  - $0 = m^a \tilde{X}_a^{\mathrm{h}}$

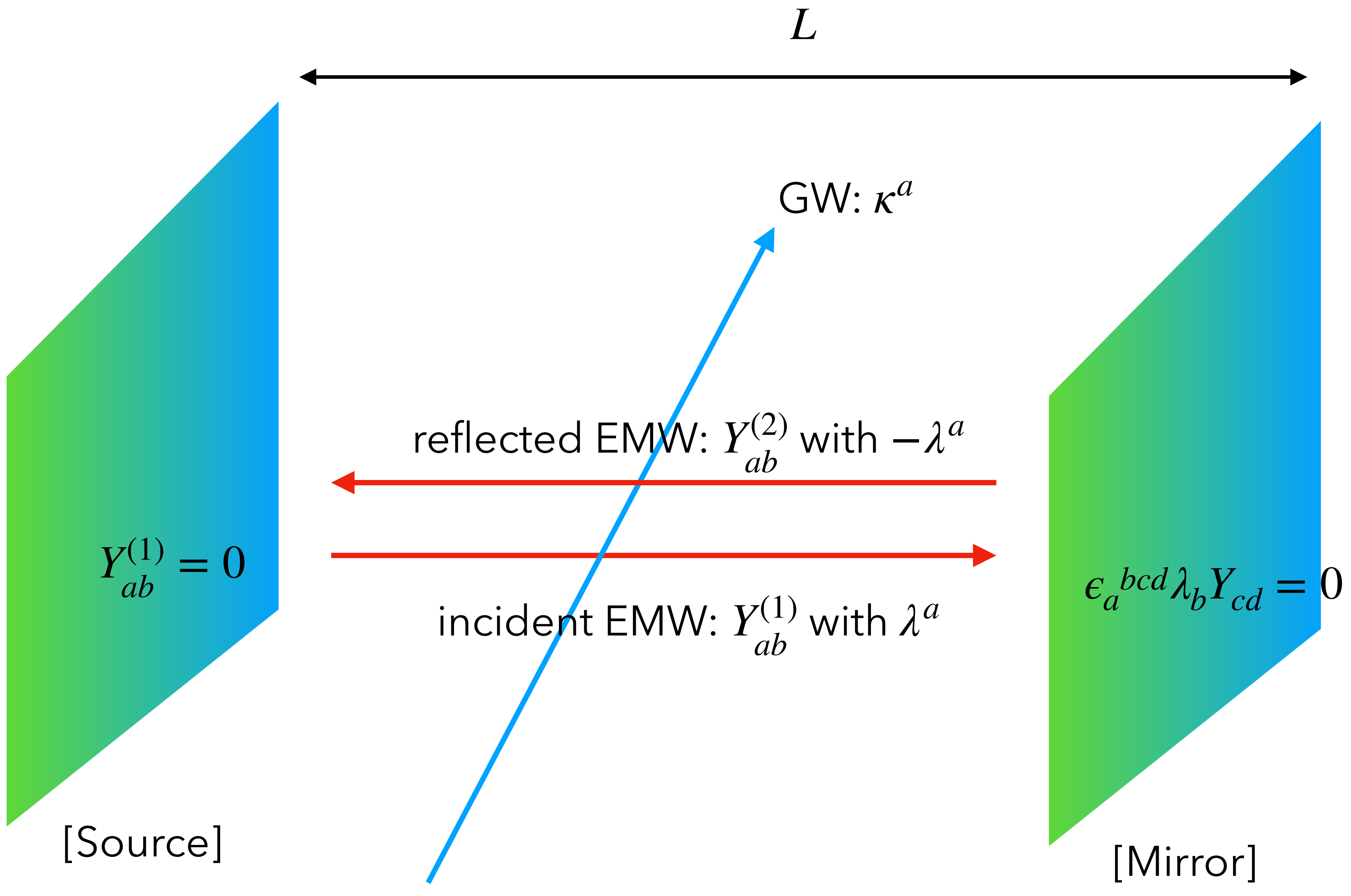
Boundary Condition at Source

- Perturbed EM field
  - ${}^{\epsilon}F_{ab} = \epsilon F_{ab} + \epsilon^2 Y_{ab} + O\left(\epsilon^3\right)$
  - $Y_{ab} = 2\nabla_{[a}X_{b]}$
- Boundary condition
  - $Y_{ab} = Y^{\mathrm{h}}_{ab} + Y^{\mathrm{p}}_{ab} = 0$ at source ($\vec{x} = 0$)
- Homogeneous solution
  - $Y^{\mathrm{h}}_{ab} = -\tilde{Y}^{\mathrm{p}}_{ab} e^{\mathrm{i}P} e^{\mathrm{i}Q} e^{\mathrm{i}\Delta}$
  - $\Delta^{\mathrm{s}}\left(\vec{x}\right) \equiv \left(\omega_{\mathrm{h}}\mu - \omega_{\mathrm{g}}\kappa - \omega_{\mathrm{e}}\lambda\right) \cdot \vec{x}$
  - $\omega_{\mathrm{h}} \equiv \omega_{\mathrm{g}} + \omega_{\mathrm{e}}$
  - $\mu^a \equiv \dfrac{\omega_{\mathrm{g}}}{\omega_{\mathrm{h}}} \{\kappa^a - (\kappa \cdot \lambda)\lambda^a\} + \left(\mu \cdot \lambda\right)\lambda^a$
- General solution
  - $Y_{ab} = \tilde{Y}^{\mathrm{p}}_{ab}\left(1 - e^{\mathrm{i}\Delta^{\mathrm{s}}}\right) e^{\mathrm{i}P} e^{\mathrm{i}Q}$



Boundary Condition at Mirror

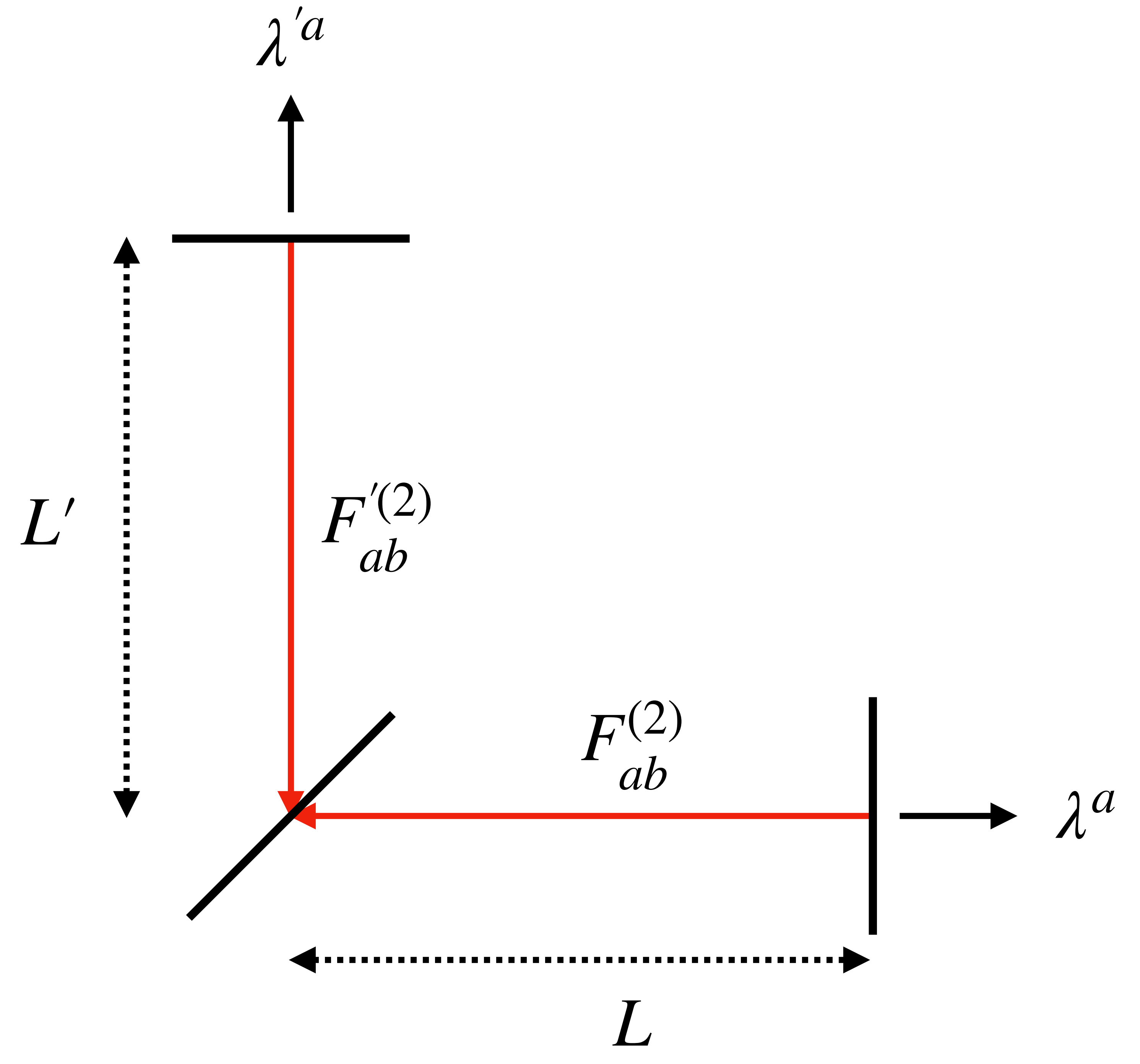
- Perturbations for two waves
  - $Y^{(1)}_{ab}$: incident wave perturbation
  - $Y^{(2)}_{ab}$: reflected wave perturbation
  - $Y_{ab} = Y^{(1)}_{ab} + Y^{(2)}_{ab}$
- Boundary condition
  - $\epsilon_a{}^{bcd}\lambda_b Y_{cd} = 0$ at mirror ($\vec{x} = L\lambda$)
- Solution of $Y^{(2)}_{ab}$
  - $\epsilon_a{}^{bcd}\lambda_b Y^{(2)}_{cd} = \epsilon_a{}^{bcd}\lambda_b \left\{ \tilde{Y}^{\mathrm{p}(1)}_{cd} \left(1 - e^{\mathrm{i}\Delta^{\mathrm{m}}}\right) + \tilde{Y}^{\mathrm{p}(2)}_{cd} \left(1 - e^{\mathrm{i}\Delta^{\mathrm{m}}} e^{2\mathrm{i}\omega_{\mathrm{e}} L}\right) \right\} e^{\mathrm{i}P} e^{\mathrm{i}Q}$
  - $\Delta^{\mathrm{m}}\left(\vec{x}\right) \equiv \left(\omega_{\mathrm{h}}\mu - \omega_{\mathrm{g}}\kappa + \omega_{\mathrm{e}}\lambda\right) \cdot \left(\vec{x} - L\lambda\right)$



Interfreometer

Background

- Ansatz
  - $\tilde{E}_a = \tilde{E}'_a = \mathscr{E} p_a e^{\mathrm{i}\delta_\mathrm{e}}$ where $p^a \equiv \epsilon^a{}_{bc}\lambda^b\lambda'^c$
- Received EMW
  - $F^\mathrm{r}{}_{ab}(t,0) = F^{(2)}_{ab}(t,0) + \mathscr{R}F'^{(2)}_{ab}(t,0)$
  - $E^\mathrm{r}_a = F^\mathrm{r}_{ab}n^b \qquad B^\mathrm{r}_a = \frac{1}{2}\epsilon^{bc}{}_a F^\mathrm{r}_{bc}$
- Intensity at dark port
  - $I^\mathrm{r} = \frac{1}{4\pi}\epsilon^{ab}{}_c E^\mathrm{r}_a B^\mathrm{r}_b \lambda^c = 4\mathscr{I}\left(1 + \cos\left(2\mathrm{i}\omega_\mathrm{e}(L - L')\right)\right)$
  - $2\mathrm{i}\omega_\mathrm{e}(L - L') = \pi + \alpha$ where $|\alpha| \ll 1$



Perturbation of Dark Port Intensity

- High frequency limit
  - We only take leading order terms proportional to $O\left(\omega_{\mathrm{e}}/\omega_{\mathrm{g}}\right)$.
- Incomplete interim results

$$\frac{\delta I^{\mathrm{r}}}{\mathscr{I}} = 4\Re\left[\frac{\omega_{\mathrm{e}}}{-\mathrm{i}\omega_{\mathrm{g}}}\tilde{h}_{ab}e^{-\mathrm{i}\omega_{\mathrm{g}}t}\left\{\frac{\lambda^a\lambda^b}{1+\cos\theta}\left(1-e^{\mathrm{i}\Delta^{\mathrm{m}}}\right)+\frac{\lambda^a\lambda^b}{1-\cos\theta}\left(\sin\left(\omega_{\mathrm{e}}\left(L+L'\right)\right)+e^{\mathrm{i}\Delta^{\mathrm{m}}}\right)\right.\right.$$

- $$\left.\left.-\frac{\lambda^a\lambda^b}{1+\cos\theta}\left(1-e^{\mathrm{i}\Delta^{\mathrm{m}}}\right)+\frac{\lambda^a\lambda^b}{1-\cos\theta}\left(\sin\left(\omega_{\mathrm{e}}\left(L+L'\right)\right)-e^{\mathrm{i}\Delta^{\mathrm{m}}}\right)\right\}\right]$$